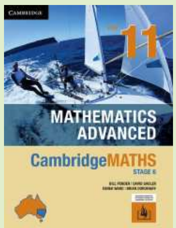
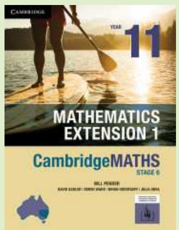
NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS ADVANCED (INCORPORATING EXTENSION 1) YEAR 11 COURSE

Topic summary and exercises:

With references to





Ⓐ **Algebric Techniques & Coordinate Geometry**

Name: ....

Initial version by H. Lam, 2018 (Algebraic Techniques), 2016 (Coordinate Geometry). Last updated February 11, 2024.
Based on the work from the legacy syllabuses by R. Trenwith, 1995–2010, subsequently maintained by H. Lam, 2011-18.
Additional editing by I. Ham and M. Ho, 2019-2020, and A. Sun in late 2020.
Various corrections by students & members of the Mathematics Departments at North Sydney Boys and Normanhurst Boys High Schools.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at `http://www.flaticon.com`, used under CC BY 2.0.

Symbols used

- Beware! Heed warning.
- Mathematics Advanced content.
- Mathematics Extension 1 exclusive content.
- Literacy: note new word/phrase.
- Facts/formulae to memorise.
- On the course Reference Sheet.
- ICT usage
- Enrichment content. Broaden your knowledge!

$\mathbb{N}$ the set of natural numbers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

MA11-1 uses algebraic and graphical techniques to solve, and where appropriate, compare alternative solutions to problems

MA11-2 uses the concepts of functions and relations to model, analyse and solve practical problems

Syllabus subtopics

MA-F1 Working with Functions

MA-F2 Graphing Techniques

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *Cambridge Year 11 3 Unit* (Pender, Sadler, Shea, & Ward, 1999) will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Contents

Part I

Algebraic Techniques

Section 1

Expansions, factorisations and algebraic fractions

1.1 Expansions and simplification

Learning Goal(s)

Knowledge
Stage 5 algebra

Skills
Factorisations and expansions

Understanding
The need to be able to perform expansions or factorisations to further simplify expressions

By the end of this section am I able to:

1.1 Perform binomial expansions

1.2 Can apply four standard methods of factorisation

1.3 Manipulate complex algebraic expressions involving algebraic fractions

Only the most challenging questions are provided here. The rest are omitted for brevity and should be assumed knowledge.

Laws/Results

Square of a sum/difference

$$(a \pm b)^2 = \ldots\ a^2 + 2ab + b^2 \ldots$$

Difference of squares

$$(a - b)(a + b) = \ldots\ a^2 - b^2 \ldots$$

Example 1

Expand and fully simplify $(a + b + c)^2$ by rewriting as $\big(a + (b + c)\big)^2$.

Answer: $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$

1.2 Factorisations

Simply, to rewrite a sum of two or more terms into a ....multiplication........ of two or more terms.

Laws/Results

Four elementary methods of factorisation, which can be nested within each other:

Highest common factor

$$5a^2b + 10ab^2 = 5ab(a+2b)$$

Difference of squares

$$a^2 - b^2 = (a-b)(a+b)$$

Quadratic trinomial

$$a^2 - 4a - 77 = (a-11)(a+7)$$

Grouping in pairs

$$3x - 3y + ay - ax = x(3-a) - y(3-a)$$

$$= (x-y)(3-a)$$

Gentle reminder

For further revision, see the *A complete course on factorisations*, located at Lowe and Lam (2010)

Exercises

1. Simplify

(a) $2a - 3b - a + 2b$ (b) $5x^2 - 3x + x - x^2 + 2x$ (c) $a - (0.1)a$

2. Simplify

(a) $3a^4 \times 2a$
(b) $5a^3 \times 3a^2b^2$
(c) $(-3x) \times (-4x)$
(d) $(3a^2)^3$
(e) $(-2a^2b)^5$
(f) $16x^8 \div 2x^2$
(g) $\dfrac{6x^3y^2z}{9x^2y^4z}$
(h) $(pq^3)^2 \times p^3q \div (pq)^3$
(i) $x(2x+1)^3 \times x^4(2x+1)$
(j) $t \times \dfrac{1}{t}$
(k) $\sqrt{x} \times \sqrt{x}$
(l) reciprocal of $a + b$

3. (a) From $3x - 2y + 1$ take $x - y + 1$.

(b) What must be added to $2m^2 - m + 1$ to give $5m^2 - 3m$?

(c) How many times must p be subtracted form p^3 to give zero?

(d) What is the product of x and

i. half the reciprocal of x? ii. the reciprocal of half x?

4. Expand, simplifying where necessary:

(a) $3(2a - 5)$
(b) $3ab(2a + 5b)$
(c) $-(1 - x)$
(d) $(2x - 1)(x + 3)$
(e) $(3a - 2)^2$
(f) $(3q^2 + 2)^2$
(g) $(1 - 2x)(1 + 2x)$
(h) $(w^3 + 2v^2)(w^3 - 2v^2)$
(i) $2(a - 3) - (1 - 2a)$
(j) $1 - (x - 1)^2$
(k) $(a+b)^2 - (a-b)(a+b)$
(l) $3(v - 2)^2$
(m) $\left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right)$
(n) $a(a + 2)(3a - 4)$
(o) $(a + b)(a - 2b + 1)$
(p) $(a - b)(a^2 + ab + b^2)$
(q) $(x - 2)(x^2 + 3x - 1)$
(r) $(a - 1)(a - 2)^2$
(s) $(a - 2)^3$
(t) $(2a + 1)^4$
(u) $\left(x + \dfrac{1}{x}\right)^2$

5. (a) Find an expansion for $(a + b + c)^2$ by first rewriting it as $\left[(a + b) + c\right]^2$. Try to guess the expansion for $(a + b + c + d)^2$.

(b) A square has side length a. The length of the square is increased by b where $0 < b < a$, while the width decreased by b. Does the area increase or decrease? By how much?

(c) In a right angled triangle, the length of the hypotenuse is $(x + 4)$ cm and the length of one of the short sides is $(x - 4)$ cm. Find an expression, in simplest form, for the length of the other short side.

6. Factorise *fully*:

(a) $6a^2b^3 - 9ab^5$
(b) $a(b+1) - b(b+1)$
(c) $9a^2 - 16b^2$
(d) $-4x + 6$
(e) $x^2 - 3x - 54$
(f) $3x^2 - 8x - 3$
(g) $12x^2 + x - 6$
(h) $x^3 + x^2 + x + 1$
(i) $x^3 + x^2 - x - 1$
(j) $x^4 - 1$
(k) $4x^2 - 16y^2$
(l) $x^2 - 6x + 9$
(m) $4x^2 + 12xy + 9y^2$
(n) $\ell^4 - 4mn\ell^2 + 4m^2n^2$
(o) $6a^2 + 7ab - 3b^2$
(p) $x^2 - 2\frac{1}{4}$
(q) $\frac{a^2}{9} - \frac{4}{25}$
(r) $(a+b)^2 - c^2$
(s) $a^2 - (b+c)^2$
(t) $(a+b)^2 - (c-d)^2$
(u) $x^2 - y^2 - x - y$
(v) $x^3 + 6x^2 + 9x$
(w) $-x^2 + 2x + 15$
(x) $x^4 - x^2 - 12$
(y) $a(a-b)^2 - ac^2$
(z) $\left(x^2 - y^2\right)^2 - (x-y)^4$

7. Show that $(x+1)^2 - (x-1)^2 = 4x$ by

(a) expanding and simplifying, and
(b) factorising, without expanding.

8. Expand and factorise $1 - x\Big(1 - x(1-x)\Big)$.

9. Factorise fully (without expanding):

(a) $3(x+1)^2 + 2x(x+1)$
(b) $x^4 - x^2(2x-1)$
(c) $x^2(x-1)^3 - 2x(x-1)^2$

10. If $f(n) = n(n+1)(n+2)$, express $f(n) + f(n+1)$ in factored form.

Further exercises

Ⓐ **Ex 1B**
- Q3-7, 9

Ⓐ **Ex 1C**
- Q1-7

(x1) **Ex 1B**
- Q7, 8, 9, 10

(x1) **Ex 1C**
- Q1-9

Answers

1. (a) $a-b$ (b) $4x^2$ (c) $\frac{9a}{10}$ or $0.9a$ **2.** (a) $6a^5$ (b) $15a^5b^2$ (c) $12x^2$ (d) $27a^6$ (e) $-32a^{10}b^5$ (f) $8x^6$ (g) $\frac{2x}{3y^2}$ (h) p^2q^4 (i) $x^5(2x+1)^4$ (j) 1 (k) x (l) $\frac{1}{a+b}$ note: NOT $\frac{1}{a}+\frac{1}{b}$ **3.** (a) $2x-y$ (b) $3m^2-2m-1$ (c) p^2 times (d) i. $\frac{1}{2}$ ii. 2 **4.** (a) $6a-15$ (b) $6a^2b+15ab^2$ (c) $x-1$ (d) $2x^2+5x-3$ (e) $9a^2-12a+4$ (f) $9q^4+12q^2+4$ (g) $1-4x^2$ (h) w^6-4v^4 (i) $4a-7$ (j) $2x-x^2$ (k) $2b^2+2ab$ (l) $3v^2-12v+12$ (m) x^2-3 (n) $3a^3+2a^2-8a$ (o) $a^2-ab+a+b-2b^2$ (p) a^3-b^3 (q) x^3+x^2-7x+2 (r) a^3-5a^2+8a-4 (s) $a^3-6a^2+12a-8$ (t) $16a^4+32a^3+24a^2+8a+1$ (u) $x^2+2+\frac{1}{x^2}$ **5.** (a) $a^2+b^2+c^2+2ab+2ac+2bc$; $a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$. (b) decreases by b^2 (c) $4\sqrt{x}$ cm **6.** (a) $3ab^3\left(2a-3b^2\right)$ (b) $(b+1)(a-b)$ (c) $(3a-4b)(3a+4b)$ (d) $-2(2x-3)$ (e) $(x-9)(x+6)$ (f) $(3x+1)(x-3)$ (g) $(4x+3)(3x-2)$ (h) $(x+1)(x^2+1)$ (i) $(x+1)^2(x-1)$ (j) $(x-1)(x+1)(x^2+1)$ (k) $4(x-2y)(x+2y)$ (l) $(x-3)^2$ (m) $(2x+3y)^2$ (n) $\left(\ell^2-2mn\right)^2$ (o) $(3a-b)(2a+3b)$ (p) $\left(x-\frac{3}{2}\right)\left(x+\frac{3}{2}\right)$ (q) $\left(\frac{a}{3}-\frac{2}{5}\right)\left(\frac{a}{3}+\frac{2}{5}\right)$ (r) $(a+b+c)(a+b-c)$ (s) $(a+b+c)(a-b-c)$ (t) $(a+b+c-d)(a+b-c+d)$ (u) $(x+y)(x-y-1)$ (v) $x(x+3)^2$ (w) $-(x-5)(x+3)$ (x) $(x-2)(x+2)(x^2+3)$ (y) $a(a-b+c)(a-b-c)$ (z) $4xy(x-y)^2$ **7.** "Show" question. **8.** $(1-x)\left(1+x^2\right)$ **9.** (a) $(x+1)(5x+3)$ (b) $x^2\left(x-1\right)^2$ (c) $x(x-1)^2(x+1)(x-2)$ **10.** $(n+1)(n+2)(2n+3)$

1.3 Algebraic Fractions

Important note

- Obtain a ......common...... ......denominator...... prior to adding.
- Only multiply up by as much as required.

Example 2

Fully simplify:

(a) $$\frac{1}{x-4}-\frac{1}{x}$$

(b) $$\frac{2}{x^2-x}-\frac{5}{x^2-1}$$

Answer: (a) $\frac{4}{x(x-4)}$ (b) $\frac{2-3x}{x(x-1)(x+1)}$

Exercises

1. Simplify fully. Also state any values of pronumeral(s) for which the simplification is not valid.

(a) $\dfrac{4a+6}{4}$

(b) $\dfrac{4a+6}{6a+9}$

(c) $\dfrac{x^2}{5x-x^2}$

(d) $\dfrac{x^2-2x-3}{x+1}$

(e) $\dfrac{x+1}{x^2-2x-3}$

(f) $\dfrac{1-x}{x-1}$

(g) $\dfrac{4a-8}{12-6a}$

(h) $\dfrac{a^2-4}{a+2}$

(i) $\dfrac{a^2+a-6}{a^2-3a+2}$

(j) $\dfrac{1-b^2}{b^3-1}$

(k) $\dfrac{x^4-5x^2+4}{x^2-x-2}$

(l) $\dfrac{x^3+2x^2-4x-8}{x^2+4x+4}$

2. Write as a single fraction in simplest form.

(a) $\dfrac{a}{2}+\dfrac{2a}{3}$

(b) $\dfrac{m-1}{2}-\dfrac{2m-3}{5}$

(c) $x+\dfrac{1}{x}$

(d) $a-\dfrac{a+b}{3}$

(e) $\dfrac{4}{a}-\dfrac{3}{b}$

(f) $\dfrac{1}{a^2}+\dfrac{2}{a}$

(g) $\dfrac{a+1}{2a}-\dfrac{a-2}{3a}$

(h) $\dfrac{1}{(x-3)^2}+\dfrac{1}{x-3}$

(i) $\dfrac{3}{a^3b}-\dfrac{2}{a^2b^4}$

(j) $\dfrac{2}{x-2}+\dfrac{3}{x+1}$

(k) $\dfrac{2x}{x+3}-\dfrac{x-2}{x+1}$

(l) $\dfrac{1}{x^2+x}+\dfrac{1}{x^2-x}$

(m) $\dfrac{x}{(x-2)(x+2)}+\dfrac{x+1}{(x-2)(x-1)}$

(n) $\dfrac{2}{y+2}-\dfrac{1}{y+3}+\dfrac{5}{y-1}$

(o) $\dfrac{k}{k+1}+\dfrac{1}{k^2+3k+2}$

(p) $\dfrac{1}{x^2-x-2}-\dfrac{1}{x^2+5x+4}-\dfrac{1}{x^2+2x-8}$

(q) $\dfrac{a^2}{a^2+3a+2}-\dfrac{2a}{a+2}$

3. Simplify fully:

(a) $\dfrac{4x^2}{3y^2}\times\dfrac{6y}{15x^4}$

(b) $\dfrac{3a^3}{7b^2}\div\dfrac{9a}{7b}$

(c) $\dfrac{2}{a}\div a$

(d) $\dfrac{x+1}{2}\times\dfrac{4x}{(x+1)^2}$

(e) $\dfrac{1}{x+2}\times\dfrac{4x+8}{3}$

(f) $\dfrac{9}{x^3+64}\div\dfrac{6}{x+4}$

(g) $\dfrac{x+y}{a-b}\times\dfrac{b-a}{y+x}$

(h) $\dfrac{a^2-a}{6a^3+6a^2}\div\dfrac{a^2-1}{8a}$

(i) $\dfrac{a^2-2a-3}{a^2+3a}\times\dfrac{3a^2+18a+27}{a^2-9}$

4. x is the smallest of three consecutive integers.

(a) Find as a single fraction in simplest form, an expression for the sum of the reciprocals of these integers.

(b) Three fractions are formed by dividing each of these integers by the integer following it. Find an expression, in simplest form, for the product of these fractions.

5. Find the reciprocal of $\dfrac{1}{a} + \dfrac{1}{b}$.

6. Simplify:

(a) $\dfrac{\frac{1}{m} + \frac{1}{n}}{m+n}$ (b) $\dfrac{1}{1 - \frac{m}{n}} + \dfrac{1}{1 - \frac{n}{m}}$ (c) $\dfrac{1 - \frac{2}{t+1}}{t - \frac{2}{t+1}}$

7. (a) If $f(n) = n(n+1)(n+2)$, simplify $\dfrac{f(n)}{f(n+1)}$.

(b) If $f(n) = \dfrac{n^2}{n-1}$, prove that $f\left(\dfrac{t}{t-1}\right) = f(t)$.

Answers

1. (a) $\frac{2a+3}{2}$ (b) $\frac{2}{3}$ $\left[a \neq -\frac{3}{2}\right]$ (c) $\frac{x}{5-x}$ $[x \neq 0, 5]$ (d) $x - 3$ $[x \neq 1]$ (e) $\frac{1}{x-3}$ $[x \neq -1, 3]$ (f) -1 $[x \neq 1]$ (g) $-\frac{2}{3}$ $[a \neq 2]$ (h) $a - 2$ $[a \neq -2]$ (i) $\frac{a+3}{a-1}$ $[a \neq 1, 2]$ (j) $-\frac{b+1}{b^2+b+1}$ $[b \neq 1]$ (k) $(x+2)(x-1)$ $[x \neq -1, 2]$ (l) $x - 2$ $[x \neq -2]$ **2.** (a) $\frac{a}{6}$ (b) $\frac{m+1}{10}$ (c) $\frac{x^2+1}{x}$ (d) $\frac{2a-b}{3}$ (e) $\frac{4b-3a}{ab}$ (f) $\frac{2a+1}{a^2}$ (g) $\frac{a+7}{6a}$ (h) $\frac{x-2}{(x-3)^2}$ (i) $\frac{3b^3-2a}{a^3b^4}$ (j) $\frac{5x-4}{(x-2)(x+1)}$ (k) $\frac{x^2+x+6}{(x+3)(x+1)}$ (l) $\frac{2}{x^2-1}$ (m) $\frac{2(x^2+x+1)}{(x-2)(x+1)(x+2)}$ (n) $\frac{2(3y^2+14y+13)}{(y+2)(y+3)(y-1)}$ (o) $\frac{k+1}{k+2}$ (p) $\frac{-x+5}{(x-2)(x+1)(x+4)}$ (q) $-\frac{a}{a+1}$ **3.** (a) $\frac{8}{15x^2y}$ (b) $\frac{a^2}{3b}$ (c) $\frac{2}{a^2}$ (d) $\frac{2x}{x+1}$ (e) $\frac{4}{3}$ (f) $\frac{3}{2(x^2-4x+16)}$ (g) -1 (h) $\frac{4}{3(a+1)^2}$ (i) $\frac{3(a+1)}{a}$ **4.** (a) $\frac{3x^2+6x+2}{x(x+1)(x+2)}$ (b) $\frac{x}{x+3}$ **5.** $\frac{ab}{a+b}$ (NOT $a + b$!!!) **6.** (a) $\frac{1}{mn}$ (b) 1 (c) $\frac{1}{t+2}$ **7.** (a) $\frac{n}{n+3}$

Further exercises

(A) Ex 1D

- Q1-11 last column

(X1) Ex 1D

- Q2-13 last column

Section 2

Equations

Learning Goal(s)

Knowledge
Distinguishing the type of equation

Skills
Solve the equation using the appropriate technique

Understanding
The importance of being able to formulate and solve equations

By the end of this section am I able to:

1.4 Solve linear equations and change the subject of a formula.

1.5 Solve quadratic equations using the quadratic formula and by completing the square

1.6 Solve simultaneous equations (linear and quadratic) by substitution and elimination.

2.1 Linear equations and inequalities

Laws/Results

Two standard forms of linear equations:

1. ......Gradient...... - ......intercept...... form

$$y = \underline{\quad mx + c \quad}$$

2. ......General...... form

$$\underline{\quad ax + by + c \quad} = 0$$

where a is a ......positive...... whole number, and b and c are ......whole...... numbers.

Vertical/horizontal lines

- Vertical line: $x = a$
- Horizontal line: $y = b$

Further exercises

(A) Ex 1E
- Q1-3, 6-10

(x1) Ex 1E
- Q2-4, 7, 8, 11

(x1) Ex 5A
- Q3

2.2 Quadratic equations

2.2.1 Completing the square

Example 3

Complete the square for $x^2 + 4x + 1$ and rewrite as $(x - h)^2 + k$.

(See next page for the steps box)

Steps

1. Halve the coefficient of x, and then square it.

2. Complete the square:

Exercises

1. Complete the square:

(a) x^2+6x+1 (b) x^2-2x-5 (c) a^2+4a (d) $x^2-8x+20$ (e) x^2+x+1 (f) x^2-3x-3 (g) a^2-a (h) $4x^2+4x-5$ (i) $9x^2-24x+14$ (j) $x^2+\frac{1}{2}x+1$ (k) $x^2+\frac{2}{3}x-2$ (l) $x^2-\frac{3}{4}x$

2. Factorise by completing the square:

(a) x^2+2x-8 (b) a^2+2a-1 (c) m^2-4m+1 (d) $4m^2+12m+3$

3. Complete the square, then factorise:

(a) $x^2+2bx+c$ (b) $x^2+\dfrac{b}{a}x+\dfrac{c}{a}$

Answers

1. (a) $(x+3)^2-8$ (b) $(x-1)^2-6$ (c) $(a+2)^2-4$ (d) $(x-4)^2+4$ (e) $\left(x+\frac{1}{2}\right)^2+\frac{3}{4}$ (f) $\left(x-\frac{3}{2}\right)^2-\frac{21}{4}$ (g) $\left(a-\frac{1}{2}\right)^2-\frac{1}{4}$ (h) $(2x+1)^2-6$ (i) $(3x-4)^2-2$ (j) $\left(x+\frac{1}{4}\right)^2+\frac{15}{16}$ (k) $\left(x+\frac{1}{3}\right)^2-\frac{19}{9}$ (l) $\left(x-\frac{3}{8}\right)^2-\frac{9}{64}$ **2.** (a) $(x+4)(x-2)$ (b) $\left(a+1+\sqrt{2}\right)\left(a+1-\sqrt{2}\right)$ (c) $\left(m-2+\sqrt{3}\right)\left(m-2-\sqrt{3}\right)$ (d) $\left(2m+3-\sqrt{6}\right)\left(2m+3+\sqrt{6}\right)$ **3.** (a) $\left(x+b+\sqrt{b^2-c}\right)\left(x+b-\sqrt{b^2-c}\right)$ (b) $\left(x+\frac{b+\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b-\sqrt{b^2-4ac}}{2a}\right)$

Further exercises

(A) **Ex 1H**
- Q1-6

(X1) **Ex 1H**
- Q1-6

2.2.2 Methods of solution

Laws/Results

Three methods of solution:

- ....Factorisation........
- ....Completing........ the ....square........ (rarely used as it is generalised into the method below)
- ....Quadratic........ ....formula........

Gentle reminder

Difference between an *equation* and *expression*

- An *equation* has an ....equals........ sign in the question. Commences with ....solve........ .
- An *expression* does not have an ....equals........ sign in the question. Usually commences with ....simplify........ , ...expand........ or ....factorise........ .

Important note

⚠ When solving quadratic equations, always solve with ....zero........ as the subject!

Example 4

(a) Fully factorise: $5x^2 + 34x - 7$

(b) Hence solve: $5x^2 + 34x - 7 = 0$ **Answer:** $\frac{1}{5}$ or -7

Example 5

Solve: $\dfrac{2}{a+3} + \dfrac{a+3}{2} = \dfrac{10}{3}$ **Answer:** $-\frac{7}{3}$ or 3

Further exercises

(A) **Ex 1F**

- Q2-11

(X1) **Ex 1F**

- Q3-13

2.3 Simultaneous equations

Important note

Three methods of solution:

1. Graphical

- The number of solutions to a pair of equations is geometrically, the number of points of intersection of two corresponding graphs.

2. Substitution

3. Elimination

Example 6

Solve: $\begin{cases} 3x - 2y = 29 \\ 4x + y = 24 \end{cases}$

Answer: $x = 7$, $y = -4$

Example 7

Solve: $\begin{cases} y = x^2 \\ y = x + 2 \end{cases}$

Answer: $x = 2$, $y = 4$ or $x = -1$, $y = 1$

Important note

After substituting, what type of equation shows up?

Example 8

Solve: $\begin{cases} x^2 + y^2 = 53 \\ x^2 - y^2 = 45 \end{cases}$

Answer: $x = 7$, $y = \pm 2$ or $x = -7$, $y \pm 2$

2.3.1 Break-even analysis

One application of simultaneous equations is to find the break-even point in manufacturing and sales.

Fill in the spaces

- ....Cost.... ....function.... : A formula for the cost to produce a given number of items
- ....Revenue.... ....function.... : the sale price multiplied by the number of units sold

Definition 1

....Break.... - ....even.... ....point.... is when the cost function and revenue function ....intersect.... .

Example 9

(Fitzpatrick & Aus, 2018, p.125) Georgia decides to turn her jewellery-making hobby into a small business. She spends \$120 on equipment and estimates that it costs her \$8 to manufacture each item. She plans to sell the items for \$20 each.

(a) If C is her total cost and R her total revenue, in dollars, set up the cost and revenue functions for the sale of x items.

(b) Determine her break-even point.

(c) Determine her profit if she sells:

(i) 6 items

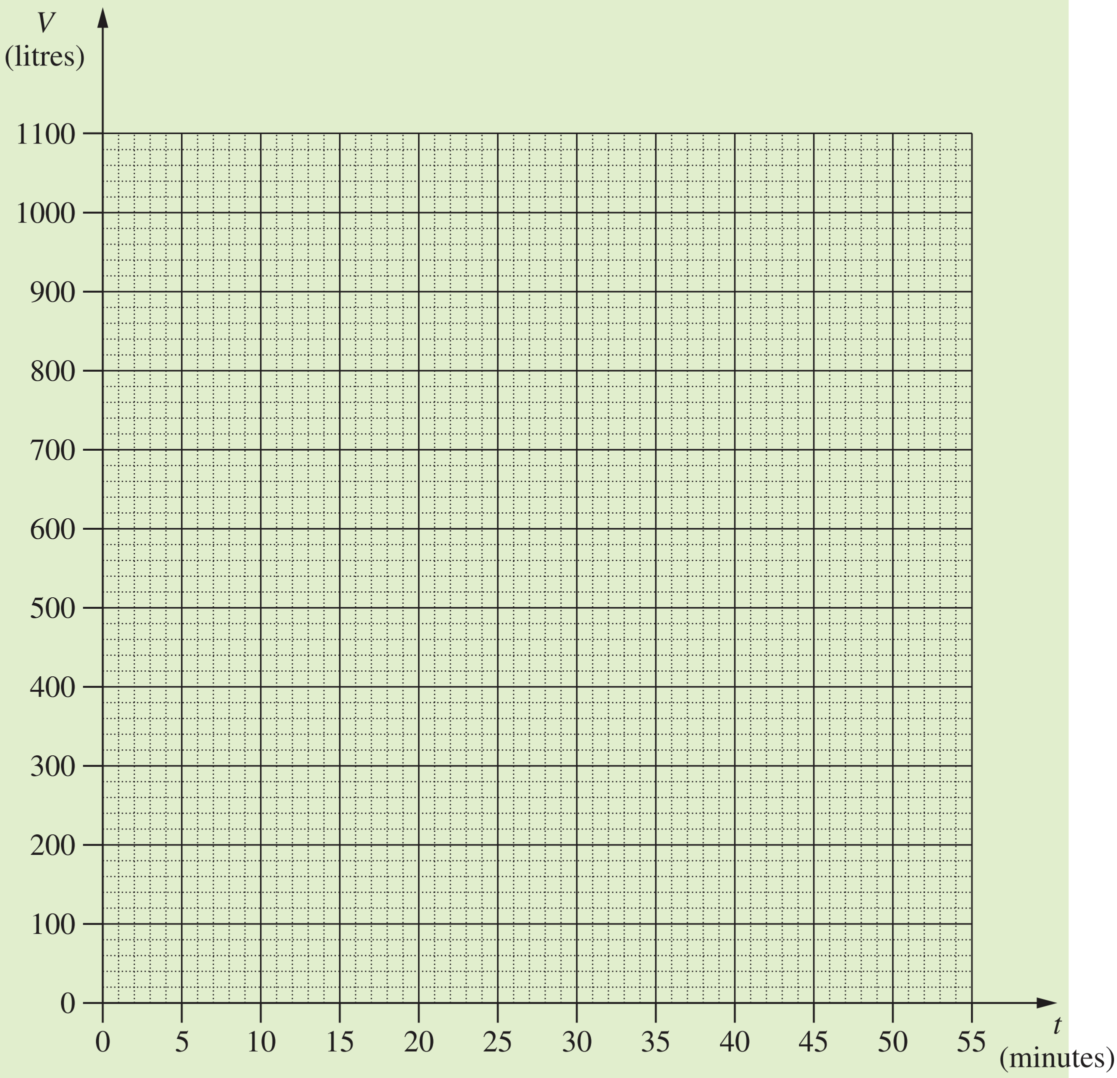
(ii) 50 items

Example 10

[2020 Adv HSC Q11] There are two tanks on a property, Tank A and Tank B. Initially, Tank A holds 1 000 litres of water and Tank B is empty.

(a) Tank A begins to lose water at a constant rate of 20 litres per minute. The volume of water in Tank A is modelled by $V = 1000 - 20t$ where V is the volume in litres and t is the time in minutes from when the tank begins to lose water. **1**

On the grid below, draw the graph of this model and label it as Tank A.



Example 11

[2020 Adv HSC Q11] (*continued from previous page*)

(b) Tank B remains empty until $t = 15$ when water is added to it at a constant rate of 30 litres per minute. **2**

By drawing a line on the grid on the previous page, or otherwise, find the value of t when the two tanks contain the same volume of water.

(c) Using the graphs drawn, or otherwise, find the value of t (where $t > 0$) when the total volume of water in the two tanks is 1000 litres. **1**

Exercises

Source: Fitzpatrick and Aus (2018, Ex 5.5), unless otherwise stated.

1. For each set of cost and revenue functions (a)-(d), find:

i. the break-even point

ii. the revenue at the break-even point

iii. the profit function, P

(a) $C = 5x + 200, R = 15x$

(b) $C = 0.5x + 100, R = 1.5x$

(c) $C = 15x + 3000, R = 45x$

(d) $C = 0.3x + 5000, R = 1.1x$

2. Julian buys a coffee cart for \$15 000. His repayments work out to be \$80 per day for the first year. He calculates that it will cost him \$2 per cup of coffee for the ingredients. He sells coffee at \$4 per cup and he sells x cups of coffee per day.

(a) Write the cost function C and revenue function R.

(b) What is his break-even point?

(c) Write the profit function P.

(d) What is his profit if he sells 100 cups per day?

3. Maya runs a market stall at the weekends, selling paintings. It costs here \$90 per day for the site. It costs her on average \$4 per painting that she sells and she sells them for an average price of \$10. If she sells x articles each day, find:

(a) the cost function C and revenue function R.

(b) her break-even point

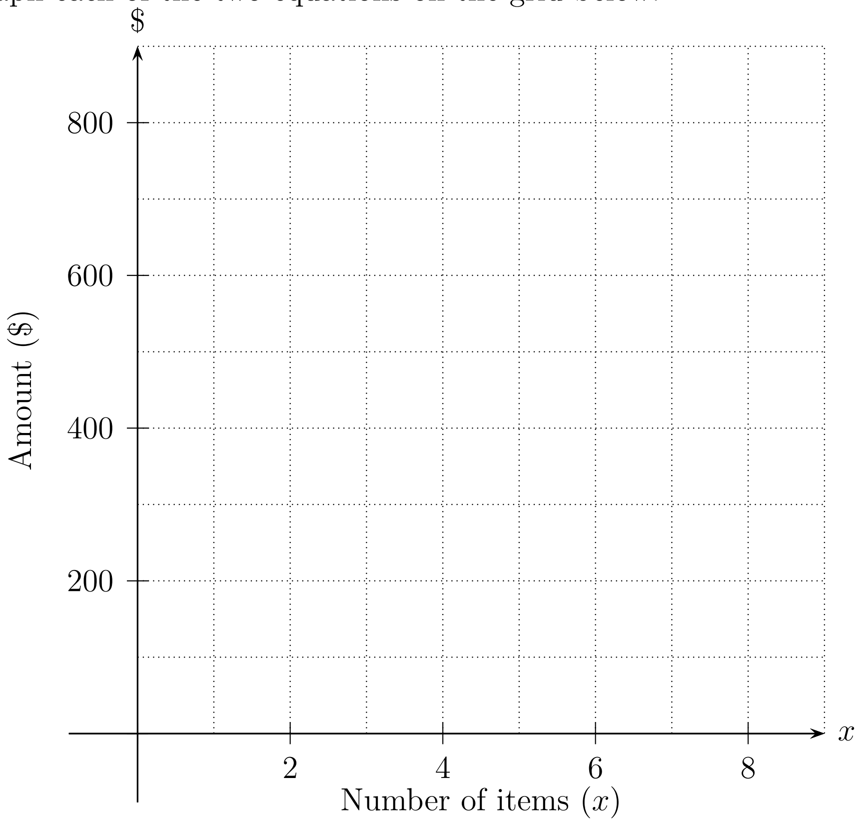
(c) the profit she will make if she sells 40 paintings in a day.

(d) One weekend, the weather is fine on the Saturday and rainy on the Sunday. On Saturday Maya sells 30 paintings, but on Sunday she only sells 10 paintings. What profit (or loss) does she make for the weekend?

4. **[2020 CSSA Adv Trial Q14]** Michael has a small manufacturing business.

The cost of manufacturing is given by the equation $C = 50x + 200$ and the income earned is given by the equation $I = 100x$, where x is the number of items that the business has manufactured.

(a) Graph each of the two equations on the grid below. **2**



(b) How many items need to be manufactured for the business to break even? **1**

Answers

1. (a) i. $x = 20$ ii. $R = \$300$ iii. $P = 10x - 200$ (b) i. $x = 100$ ii. $R = \$150$ iii. $P = x - 100$ (c) i. $x = 100$ ii. $R = \$4500$ iii. $P = 30x - 3000$ (d) i. $x = 6250$ ii. $R = \$6875$ iii. $P = 0.8x - 5000$ **2.** (a) $C = 2x + 80, R = 4x$ (b) $x = 40$ (c) $P = 2x - 80$ **3.** (a) $C = 4x + 90, R = 10x$ (b) $x = 15$ (c) $P = 6x - 90, x = 40, P = \$150$ (d) Saturday $x = 30, P = \$90$/ Sunday $x = 10, P = -\$30$/ Profit for the weekend: \$60 **4.** 4 items

Further exercises

(A) **Ex 1G**
- Q3-5

(X1) **Ex 1G**
- Q3-5

Section 3

Surd laws

Learning Goal(s)

Knowledge
Simplifying surds

Skills
Rewriting surds in lowest base and rationalising

Understanding
The difference between *simplifying* and *rationalising* surds

By the end of this section am I able to:
1.7 Use surd laws

3.1 Basic arithmetic of surds

Theorem 1

Fully simplify surds by finding their ......greatest...... ......perfect...... ......square...... factor.

Example 12

Fully simplify: $\sqrt{72}-\sqrt{50}+\sqrt{12}$ **Answer:** $\sqrt{2}+2\sqrt{3}$

Example 13

Expand and fully simplify: $\left(\sqrt{15}-\sqrt{6}\right)^2$ **Answer:** $21-6\sqrt{10}$

Theorem 2

Rewrite surds with a ....rational.... denominator by multiplying numerator and denominator by the ....conjugate.... of the denominator.

Example 14

Fully simplify: $\dfrac{1}{2\sqrt{3}-3\sqrt{2}}$.

Answer: $-\frac{2\sqrt{3}+3\sqrt{2}}{6}$

3.2 Equality of surds

Theorem 3

Two surds $a + b\sqrt{c}$ and $x + y\sqrt{c}$ are equal iff $a = x$ and $b = y$.

Example 15

Find the value of a and b if $\dfrac{2}{\sqrt{5}+1} = a + b\sqrt{5}$.

Answer: $a = -\frac{1}{2}, b = \frac{1}{2}$

Exercises

Acknowledgement: Portions taken from Grove (2010, Ex 2.23)

1. Find the values of a and b if

(a) $\dfrac{3}{2\sqrt{5}} = \dfrac{\sqrt{a}}{b}$

(b) $\dfrac{\sqrt{3}}{4\sqrt{2}} = \dfrac{a\sqrt{6}}{b}$

(c) $\dfrac{2\sqrt{7}}{\sqrt{7}-4} = a + b\sqrt{7}$

(d) $\dfrac{\sqrt{2}+3}{\sqrt{2}-1} = a + \sqrt{b}$

2. Show that $\dfrac{\sqrt{2}-1}{\sqrt{2}+1} + \dfrac{4}{\sqrt{2}}$ is rational.

3. If $x = \sqrt{3} + 2$, fully simplify:

(a) $x + \dfrac{1}{x}$

(b) $\left(x + \dfrac{1}{x}\right)^2$

(c) $x^2 + \dfrac{1}{x^2}$

4. If $2 + \dfrac{1}{x} = \sqrt{3}$ where $x \neq 0$, find the exact value of x (with a rational denominator).

Answers

1. (a) $a = 45, b = 10$ (b) $a = 1, b = 8$ (c) $a = -\frac{14}{9}, b = -\frac{8}{9}$ (d) $a = 5, b = 32$ **2.** Show **3.** (a) 4 (b) 16 (c) 14 **4.** $-\left(\sqrt{3}+2\right)$

Further exercises

(A) **Ex 2C**
- Q1-8 last column

(A) **Ex 2D**
- Q1-11 last 2 columns

(A) **Ex 2E**
- last 2 columns

(X1) **Ex 2C**
- Q1-8 last column

(X1) **Ex 2D**
- Q1-13 last 2 columns

(X1) **Ex 2E**
- last 2 columns

Part II

Coordinate Geometry

Section 4

Points and intervals

Learning Goal(s)

Knowledge
The distance and midpoint formula in the Cartesian plane

Skills
Finding the distance and midpoint of an interval

Understanding
The derivation of the distance and midpoint formulas

By the end of this section am I able to:

1.8 Distance formula between two points in the Cartesian Plane

1.9 The midpoint formula

4.1 Review of formulae

Laws/Results

The distance formula to calculate the distance between two points (x_1, y_1) and (x_2, y_2):

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Derivation of formula:

Laws/Results

The midpoint formula to calculate the midpoint between two points (x_1, y_1) and (x_2, y_2):

$$M(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Derivation of formula:

Example 16

The interval joining $A(3, -7)$ and $B(-6, 2)$ is a diameter of a circle. Find the centre and radius of the circle.

Answer: $C\left(-\frac{3}{2}, -\frac{5}{2}\right)$, $r = \frac{9}{2}\sqrt{2}$

Further exercises

(2) Ex 6A

- Q1-2 last column
- Q3-10
- Q16-17

Section 5

Gradient

Learning Goal(s)

Knowledge
The relation between parallel and perpendicular lines

Skills
Determine the type of quadrilateral by examining the gradient of its sides and diagonals

Understanding
The relation between the gradient and the angle of inclination

By the end of this section am I able to:

1.10 Calculate the gradient of an interval

1.11 Examine and use the relationship between the angle of inclination of a line or tangent θ with the positive x-axis, and the gradient m of that line or tangent, and establish that $\tan\theta = m$

1.12 Understand and use the fact that parallel lines have the same gradient and that two lines with gradient m_1 and m_2 respectively are perpendicular if and only if $m_1 m_2 = -1$

1.13 Test for the special quadrilaterals in the Cartesian plane using distance, midpoint and gradient calculations

5.1 Review of formulae

Laws/Results

The gradient formula to calculate the gradient between two points (x_1, y_1) and (x_2, y_2):

$$m = \frac{y_2 - y_2}{x_2 - x_1}$$

Derivation of formula:

Laws/Results

The angle of inclination θ of the line and the positive direction of the x axis:

$$m = \tan\theta$$

Derivation of formula:

Laws/Results

Parallel lines have the ....same.... gradient.

Laws/Results

The gradient of two perpendicular lines are ....negative.... ....reciprocals....:

$$m_1 = -\frac{1}{m_2} \qquad \text{or} \qquad m_1 m_2 = -1$$

Example 17

Show that the points $A(-2, 5)$, $B(1, 3)$ and $C(7, -1)$ are collinear

Example 18

[Ex 5B Q16] Answer the following for the points $W(2, 3)$, $X(-7, 5)$, $Y(-1, -3)$ and $Z(5, -1)$:

(a) Show that WZ is parallel to XY.

(b) Find the lengths WZ and XY. Hence deduce the type of quadrilateral $WXYZ$.

(c) Show that that the diagonals WY and XZ are perpendicular.

Example 19

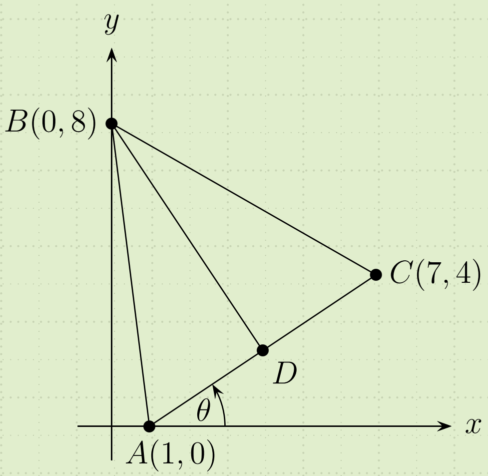
[Ex 5B Q18]

(a) $A(1,4)$, $B(5,0)$ and $C(9,8)$ form the vertices of a triangle. Find the coordinates of P and Q if they divide the sides AB and AC respectively in the ratio $1:3$.

(b) Show that PQ is parallel to BC and is one quarter of its length.

Example 20

The points A, B and C have coordinates $(1,0)$, $(0,8)$ and $(7,4)$, and the angle between AC and the x axis is θ.



(a) Find the gradient of the line AC and hence determine θ to the nearest degree. **2**

(b) Find the equation of AC. **2**

(c) Find the coordinates of D, the midpoint of AC. **2**

(d) Show that $AC \perp BD$. **1**

(e) What type of triangle is ABC? Show full reasoning. **2**

(f) Find the area of this triangle. **2**

(g) Write down the coordinates of the point E such that $ABCE$ is a rhombus. **2**

Further exercises

(2) **Ex 6B**
- Q7-21 odd #

(x1) **Ex 5B**
- Q7-21 odd #

Section 6

Equation of straight line

Learning Goal(s)

Knowledge
The different forms of straight lines

Skills
Finding the equation of straight lines

Understanding
Applying the appropriate form of a straight line to find its equation

By the end of this section am I able to:

1.14 Know and use the following forms of a straight line: gradient-intercept form $y = mx + c$, and general form $ax + by + c = 0$

1.15 Derive the equation of a straight line passing through two points (x_1, y_1) and (x_2, y_2) by first calculating its gradient m using the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$

1.16 Derive the equation of a straight line passing through two points (x_1, y_1) and (x_2, y_2) by first calculating its gradient m using the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$

1.17 Determine if three distinct lines are concurrent

1.18 Find the equations of straight lines, including parallel and perpendicular lines, given sufficient information

6.1 Review of formulae

Laws/Results

The **gradient-intercept form** of the equation of a line:

$$y = mx + b$$

where

- m – gradient
- b – y intercept

Laws/Results

The **general form** of the equation of a line:

$$ax + by + c = 0$$

Laws/Results

The **point-gradient formula** to find the equation of a straight line through (x_1, y_1) with a given gradient m:

$$\frac{y - y_1}{x - x_1} = m$$

Derivation of formula:

Important note

No new theory is in this section. However, expect slightly more difficult problems within textbook exercises.

Example 21

[**Ex 5D Q6**] Given the points $A(1, -2)$ and $B(-3, 4)$, find in general form the equation of:

(a) the line AB,

(b) the line through A perpendicular to AB.

Example 22

[Ex 5D Q12]

(a) On a number plane, plot the points $A(4, 3)$, $B(0, -3)$ and $C(4, 0)$.

(b) Find the equation of BC.

(c) Explain why $OACB$ is a parallelogram.

(d) Find the area of $OACB$ and the length of the diagonal AB.

Further exercises

2 **Ex 6C**
- Q5, 10-16

2 **Ex 6D**
- Q5
- Q7 last 3 columns
- Q14, 18-20

x1 **Ex 5C**
- Q6, 10-14

x1 **Ex 5D**
- Q4, 7-10, 13-18

References

Fitzpatrick, J. B., & Aus, B. (2018). *New Senior Mathematics – Advanced Course for Years 11 & 12* (3rd ed.). Pearson Education.

Grove, M. (2010). *Maths in focus: mathematics extension preliminary course* (E. Bron, Ed.). McGraw-Hill Australia Pty Ltd.

Lowe, R., & Lam, H. (2010). *A complete course on factorisation.* Normanhurst Boys High School Edmodo site. (Compiled by R. Lowe. Typeset in 2010 and edited in 2013 by H. Lam)

Pender, W., Sadler, D., Shea, J., & Ward, D. (1999). *Cambridge Mathematics 3 Unit Year 11* (1st ed.). Cambridge University Press.